What is Noncommutative Geometry ?

How a geometry can be commutative and why mine is not

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Doing Geometry Without a Geometric Space

Theorem

Two smooth manifolds M, N are diffeomorphic if and only if their algebras of smooth functions $\mathcal{C}^\infty(M)$ and $\mathcal{C}^\infty(N)$ are isomorphic.

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This means that the algebra $\mathcal{C}^\infty(M)$ contains enough information to codify the whole geometry of the manifold:

1. Vector fields: linear derivations of $\mathcal{C}^\infty(M)$
2. Differential 1-forms: $\mathcal{C}^\infty(M)$-linear forms on vector fields
3. ...

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- *Do we really need a manifold's points to study it ?*

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Question

- *Do we really need a manifold's points to study it ?*
- *Do we really use the commutativity of the algebra $\mathcal{C}^\infty(M)$ to define the aforementioned objects ?*

Why Consider Noncommutativity?

Consider the following sentences:

- Darling I love you
- Leaving your idol
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Moral of the Story

Commutativity $\Rightarrow$ ***Loss of Information***

Commutative Geometries

Fundamental Idea of Noncommutative Geometry

*Replace $\mathcal{C}^\infty(M)$ with a **possibly** noncommutative algebra A and regard A as the algebra of functions on a "noncommutative smooth manifold".*

$$\begin{array}{ccccc} M & \longrightarrow & \mathcal{C}^\infty(M) & \longrightarrow & \textit{Vector Fields, Diff. Forms, ...} \\ & & \downarrow & & \\ ? & \longrightarrow & A & \longrightarrow & \textit{Analogous Algebraic Constructions} \end{array}$$

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- *If A is commutative we want back all the known machinery.*
- *Noncommutative manifolds do **NOT** exist.*

The previous method can be adapted to study also:

- A topological space X and the algebra $\mathcal{C}(X)$
- A measure space (X, μ) and the algebra $L^\infty(X, \mu)$
- ...

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Noncommutative Topology

C^*-Algebras and Functions

Definition

*A C^*-**algebra** is a complex algebra A endowed with a norm A and an anti-linear map $* : A \to A$ such that they satisfy some technical axioms and the following relation:*

$$\|a^* a\| = \|a\|^2 \qquad \forall a \in A$$

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Example

Let X be a topological space. The set $\mathcal{C}_b(X)$ of continuous and bounded functions $f : X \to \mathbb{C}$ is a C^*-algebra where the product is given pointwise, the norm and the involution are defined by

$$\|f\| = \sup_{x \in X} |f(x)| \qquad (f^*)(x) = \overline{f(x)}$$

The Gelfand-Najmark Correspondence

Definition

Let X be a locally compact Hausdorff topological space. The algebra $\mathcal{C}_0(X)$ of functions that ***vanish at infinity*** *is the norm closure of the functions with compact support $\mathcal{C}_c(X)$.*

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- *Let X be a locally compact Hausdorff topological space. The space $\mathcal{C}_0(X)$ is a commutative C^*-algebra.*
- *Let A be a commutative C^*-algebra. There exists a locally compact Hausdorff topological space X such that $A \simeq \mathcal{C}_0(X)$.*

How is This Possible ?

The correspondence of the previous theorem is very well known:

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- There is a bijective correspondence between maximal modular ideals $I \subseteq A$ and the quotient maps $\gamma\colon A \to A/I \simeq \mathbb{C}$. Under this identification, X is endowed with the weak-$*$ topology.
- The isomorphism is given by the Gelfand Transform

$$\Gamma\colon A \longrightarrow \mathcal{C}_0(X) \qquad a \mapsto \widehat{a}$$

where $\widehat{a}\colon X \to \mathbb{C}$ is the point evaluation $\widehat{a}(\gamma) = \gamma(a)$.

Noncommutative Topological Spaces

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- *Every* C^**-algebra homomorphism* $F\colon \mathcal{C}_0(Y) \to \mathcal{C}_0(X)$ *is the pullback map of a continuous (and proper) map* $f : X \to Y$.

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Moral of the Story

We can consider (possibly) noncommutative C^**-algebras as a generalization of the "usual" notion of topological space.*

Noncommutative Topology: Compactification

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Question

What happens if we take the maximal unitization of the C^*-algebra A ? We get the Stone-Čech compactification of X.

Definition

Let A be a ring. An A-module $\mathcal{M}$ is said to be ***projective*** *if for every surjective module morphism $\rho\colon \mathcal{N} \to \mathcal{M}$ there exists a module morphism $s\colon \mathcal{M} \to \mathcal{N}$ such that $\rho \circ s = \mathrm{id}_{\mathcal{M}}$.*

Noncommutative Topology: Vector Bundles

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Theorem (Serre-Swan)

Let X be a compact Hausdorff topological space.

- *For every vector bundle $\pi\colon E \to X$, the space of sections $\Gamma(X, E)$ is a projective finitely generated $\mathcal{C}(X)$-module.*

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- *For every projective finitely generated $\mathcal{C}(X)$-module $\mathcal{M}$, there exists a vector bundle $\pi\colon E \to X$ such that $\mathcal{M} \simeq \Gamma(X, E)$.*

Noncommutative Topology: K-Theory

- The topological K-theory of a topological space X is the group of isomorphism classes of (complex) vector bundles over X with addition given by the direct sum of vector bundles.
- Under the identification of the Serre-Swan theorem, the NC K-theory group $K_0(A)$ is defined as the space of suitable equivalence classes of projections of a C^*-algebra A.
- There is a correspondence between the commutative and the noncommutative K-theory, namely that

$$K^0(X) \simeq K_0(\mathcal{C}_0(X))$$

for a locally compact Hausdorff topological space X.

Noncommutative Dictionary

Topology	**Algebra**
Continuous Proper Map	Morphism
Homeomorphism	Automorphism
Compact	Unital
1-point Compactification	Minimal Unitization
Stone-Čech Compactification	Multiplier Algebra
Open (Dense) Subset	(Essential) Ideal
Second Countable	Separable
Connected	Trivial Space of Projections
Vector Bundle	Fin. Gen. Proj. Module
Cartesian Product	Tensor Product

Noncommutative Measure Theory

Example

Let H be a Hilbert space. The space $\mathcal{L}(H)$ of linear and bounded operators $T\colon H \to H$ is a unital C^*-algebra where the product is given by the composition, the norm is

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and the involution is $T \mapsto T^*$ where T^* is the adjoint operator.

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Theorem (Second Gelfand-Najmark Theorem)

Any C^*-algebra admits a faithful representation on a suitable Hilbert space H. Furthermore, if A is separable, H can be chosen to be separable.

- The **strong operator topology** on $\mathcal{L}(H)$ is the locally convex vector space topology induced by the family of seminorms

$$p_x \colon \mathcal{L}(H) \longrightarrow [0, +\infty[, \qquad p_x(T) = \|Tx\|$$

varying $x \in H$.

Von Neumann Algebras

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- A **Von Neumann algebra** is a SOT-closed $*$-subalgebra of $\mathcal{L}(H)$ for some Hilbert space H.

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- A **Von Neumann algebra** is a SOT-closed $*$-subalgebra of $\mathcal{L}(H)$ for some Hilbert space H.
- Since every SOT-closed set is also norm closed, every Von Neumann algebra is in particular a C^*-algebra.

The Double Commutant Theorem

The reader should note that our definition of Von Neumann algebra is not algebraic and absolute (an algebra with axioms) but is topological and depends on having a Hilbert space.

Theorem (The Double Commutant Theorem)

A C^*-subalgebra $N \subseteq \mathcal{L}(H)$ is a Von Neumann Algebra iff:

- $\mathrm{id}_H \in N$
- $N = N''$ *where* $N' = \{\, T \in \mathcal{L}(H) \mid TS = ST \quad \forall S \in N \,\}$.

Noncommutative Measurable Functions

Theorem

- *Let X be a compact Hausdorff topological space and μ a Radon measure on X. The action of $L^\infty(X, \mu)$ on the separable Hilbert space $L^2(X, \mu)$ given by multiplication makes $L^\infty(X, \mu)$ a commutative Von Neumann algebra.*

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- *Let A be a commutative Von Neumann algebra acting on a separable Hilbert space H. There exists a compact Hausdorff topological space X and a Radon measure μ on X such that $A \simeq L^\infty(X,\mu)$ as C^*-algebras.*

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Remark

In this form, this is not an equivalence of categories.

Noncommutative (Radon) Measures

Theorem (Riesz-Markov)

Let X be a locally compact Hausdorff space.

- *Given a complex Radon Measure μ on X, the map $I\colon \mathcal{C}_0(X) \to \mathbb{C}$ given by*

$$I(f) = \int_X f(x) d\mu(x)$$

is bounded and linear.

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- *Given a bounded linear functional $I \colon \mathcal{C}_0(X) \to \mathbb{C}$, there exists a unique complex Radon measure μ such that*

$$I(f) = \int_X f(x) d\mu(x) \qquad \forall f \in \mathcal{C}_0(X)$$

Open Problems

- Noncommutative Probabilities
- Noncommutative L^p-spaces
- Noncommutative Sobolev Spaces
- ...

Noncommutative Manifolds

Houston, We Have a Problem

The Gelfand-Najmark construction perfectly works for topological spaces and continuous functions: we would like to repeat the same steps for smooth manifolds and smooth functions.

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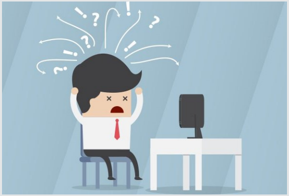
Huge Problem

$\mathcal{C}^\infty(M)$ is a Frechet space: a locally convex topological vector space that is complete with respect to a translation-invariant metric. In particular its topology is generated by a countable family of seminorms that cannot be reduced to only one norm.



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Crucial Point

Since $\mathcal{C}^\infty(M)$ is dense in the C^*-algebra $\mathcal{C}(M)$ in the sup norm, we can still follow the Gelfand-Najmark approach once we specify what in $\mathcal{C}(M)$ can be regarded as "smooth". Morally, this lack of information can be compensated by an operator D that behaves like a derivative.

Definition

Let $\mathcal{A}$ be a unital $*$-algebra dense in a C^*-algebra A, H an Hilbert space and $D\colon \mathrm{Dom}(D) \subseteq H \to H$ a densely defined self-adjoint operator. We say that the data $(\mathcal{A}, H, D)$ is a spectral triple if

- *there is a reprs. $\pi\colon A \to \mathcal{L}(H)$ on H by bounded operators*
- *D is compatible with the action of A in the sense that*
  1. *the resolvent operator $(D-\lambda)^{-1}\colon H \to H$ is compact for every λ in the resolvent set $\rho(D)$.*
  2. *the action of A preserves the domain of D and the densely defined operator $[D, a]$ extends to a bounded operator for every $a \in \mathcal{A}$*

We say that the triple is commutative if A is commutative.

A Long Story ...

The definition of a spectral triple is modelled on a canonical example from the field of Spin Geometry.
The next few slides will give you a glimpse on the construction of this canonical example.

Interlude: Spin Groups

- Consider a finite dimensional vector space V over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ endowed with a quadratic form $q\colon V \to \mathbb{K}$. The Clifford algebra $\mathcal{C}l(V, q)$ is the quotient of the tensor algebra $T(V) = \oplus_n V^{\otimes n}$ under the assumption that $v^2 = -q(v)$.

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- The Spin groups $\mathrm{Spin}(n)$ are defined as some (multiplicative) subgroup of the invertible elements in $\mathcal{C}l(\mathbb{R}^n)$.

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- The Spin groups $\mathrm{Spin}(n)$ are defined as some (multiplicative) subgroup of the invertible elements in $\mathcal{C}l(\mathbb{R}^n)$.
- One can show that for every $n \geq 1$ the spin groups are the double coverings of $\mathrm{SO}(n)$ in the sense that we have the SES:

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(n) \xrightarrow{\sigma} \mathrm{SO}(n) \longrightarrow 0$$

  Furthermore, if $n \geq 2$ they are non trivial and if $n \geq 3$ they are the universal coverings.

Interlude: Spin Manifolds

- Let M be an or. Riem. manifold. For every $p \in M$ we associate an or. orth. basis of T_pM: this gives a vector bundle F with transition functions $g_{\alpha\beta}$.

$$\begin{tikzcd} & \mathrm{Spin}(n) \arrow[d, "\sigma"] \\ U_\alpha \cap U_\beta \arrow[r, "g_{\alpha\beta}"'] & \mathrm{SO}(n) \end{tikzcd}$$

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- We say that M is a **spin manifold** if we can find a family of functions $\widetilde{g}_{\alpha\beta}$ that satisfy the cocycle conditions. In this case we denote by $\widetilde{F}$ the generated bundle. This is a topological property.

Interlude: Spinor Fields

- Given a repres. $\gamma\colon \mathsf{Spin}(n) \to \mathsf{GL}(S)$, we define the spin bundle as $\Sigma = \widetilde{F} \times_\gamma S$. Its sections are called **spinor fields** and can be regarded as γ-equivariant functions $\psi\colon \widetilde{F} \to S$.

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- The Clifford bundle is the vector bundle on M whose fibers are $\mathcal{C}l(T_pM)$ on $p \in M$. The map γ naturally induces an action

$$\gamma\colon \Gamma(T^*M \otimes \Sigma) \to \Gamma(\Sigma) \qquad \alpha \otimes \psi \mapsto \gamma(\alpha)\psi$$

such that $\gamma^2(\alpha) = -g(\alpha, \alpha)$

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$$\gamma\colon \Gamma(T^*M \otimes \Sigma) \to \Gamma(\Sigma) \qquad \alpha \otimes \psi \mapsto \gamma(\alpha)\psi$$

  such that $\gamma^2(\alpha) = -g(\alpha, \alpha)$
- There exists a connection $\nabla^S\colon \Gamma(M, \Sigma) \to \Gamma(M, T^*M \otimes \Sigma)$ compatible with the γ-action in the sense that

$$\nabla^S(\gamma(\alpha)\psi) = \gamma(\nabla\alpha)\psi + \gamma(\alpha)\nabla^S\psi$$

- The **Dirac operator** D is the composition map:

$$D\colon \Gamma(M,\Sigma) \xrightarrow{\nabla^S} \Gamma(M, T^*M \otimes \Sigma) \xrightarrow{\gamma} \Gamma(M,\Sigma)$$

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- There exists an hermitian product $\langle \cdot, \cdot \rangle$ on $\Gamma(M,\Sigma)$ with respect to which D is symmetric. If M is complete as a metric space, D is in particular essentially self-adjoint.

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- There exists an hermitian product $\langle\cdot,\cdot\rangle$ on $\Gamma(M,\Sigma)$ with respect to which D is symmetric. If M is complete as a metric space, D is in particular essentially self-adjoint.
- We denote by $L^2(M,\Sigma)$ the Hilbert space obtained by the completion of $\Gamma(M,\Sigma)$ w.r.t $\langle\cdot,\cdot\rangle$ and by $H^1(M,\Sigma)$ the completion w.r.t $\|\psi\|_H^2 = \|\psi\|_{L^2}^2 + \left\|\nabla^S\psi\right\|_{L^2}^2$. It turns out that $D\colon H^1(\Sigma) \to L^2(\Sigma)$ is linear and bounded.

- Using the Schroedinger-Lichnerowicz identity:

$$D^2 = \Delta + \frac{1}{4}R \qquad R \text{ scalar curvature}$$

one can prove that the resolvent operators go from $L^2(\Sigma)$ to $H^1(\Sigma)$ and are bounded. The Rellich-Kondrachov embedding theorem makes them compact operators.

Noncommutative (Spin) Manifolds

Connes Reconstruction Principles

- *Let M be a compact orientable Riemannian spin manifold. The triple $(\mathcal{C}^\infty(M), L^2(M,\Sigma), \overline{D})$ is a commutative spectral triple that has other seven algebraic/analytic properties.*

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- *Let M be a compact orientable Riemannian spin manifold. The triple $(\mathcal{C}^\infty(M), L^2(M,\Sigma), \overline{D})$ is a commutative spectral triple that has other seven algebraic/analytic properties.*
- *For every commutative spectral triple (A, H, D) that satisfies the before mentioned seven properties, there exists a compact orientable Riemannian spin manifold M such that (A, H, D) is the canonical spectral triple as in the previous point.*

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- *For every commutative spectral triple (A, H, D) that satisfies the before mentioned seven properties, there exists a compact orientable Riemannian spin manifold M such that (A, H, D) is the canonical spectral triple as in the previous point.*

Remark

- *The above mentioned association is not functorial.*

Noncommutative (Spin) Manifolds

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Remark

- *The above mentioned association is not functorial.*
- *We have no idea if the theorem is true if we relax the hypothesis on compactness or spin structure.*

The Role of the Dirac Operator

The Dirac operator D is crucial to capture

- the geodesic distance on the manifold by

$$d(x, y) = \sup\{|f(x) - f(y)| : f \in \mathcal{C}^\infty(M) \text{ and } \|[D, f]\| \leq 1\}.$$

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Theorem (Weyl's Law)

Let M a compact connected oriented boundary-less Riemannian spin manifold of dimension n. Consider the positive compact operator $|D| = (D^2+1)^{-\frac{1}{2}}$ defined on $L^2(M,\Sigma)$ and let λ_k be the eigenvalues listed in increasing order. Then

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$$N_{|D|}(\lambda) := \#\{\, k \in \mathbb{N} \mid \lambda_k \leq \lambda \,\} \sim \frac{\mathrm{Vol}(M)}{(4\pi)^{\frac{n}{2}}\Gamma(\frac{n}{2}+1)}\lambda^{\frac{n}{2}}$$

where Γ is the Euler Γ-function.

It's a Mess

The various additional axioms to the reconstruction theorem (that I voluntarily avoided) are related to the noncommutative formulations of:

- the orientation of the manifold (Hochschild Homology)
- the dimension of the manifold (Trace-class operators and Dixmier Ideals)
- the existence of a spin structure (Morita Equivalence of C^*-algebras)
- the absolutely continuity of the noncommutative integral
- ...

The Noncommutative Torus

The Commutative Torus

We want to apply the NC procedure to the torus $T^2 = S^1 \times S^1$.

- First of all notice that every element $F \in \mathcal{C}^\infty(T^2)$ can be regarded as a smooth function $f \colon \mathbb{R}^2 \to \mathbb{C}$ such that

$$f(x + 2\pi m, y + 2\pi n) = f(x, y) \quad \forall (m, n) \in \mathbb{Z}^2$$

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- By Fourier analysis, we can write $f \in \mathcal{C}^\infty(T^2)$ as

$$\begin{aligned} f(x, y) &= \sum_{(m,n)\in\mathbb{Z}^2} a_{m,n} e^{2\pi mix} e^{2\pi niy} \\ &= \sum_{(m,n)\in\mathbb{Z}^2} a_{m,n} U^m V^n \qquad \textit{for } U = e^{2\pi ix},\ V = e^{2\pi iy} \end{aligned}$$

  Note that $U^* = U^{-1}$, $V^* = V^{-1}$ and $UV = VU$.

The Noncommutative Torus

Definition

The Noncommutative Torus $C(T^2_\theta)$ with $\theta > 0$ is the universal C^*-algebra generated by two unitaries U_1 and U_2 such that

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We want now to show that from the C^*-algebra $\mathcal{C}(T^2_\theta)$ we can naturally build a spectral triple.

The NC Torus Spectral Triple

- We define $L^2(\mathbb{T}^2_\theta)$ as the completion of $C(\mathbb{T}^2_\theta)$ with respect to the scalar product

$$\langle a, b\rangle := \tau(a^* b) \quad \text{where} \quad \tau\left(a = \sum_{n,m\in\mathbb{Z}} a_{mn} U^m V^n\right) := a_{00}$$

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- The Dirac operator $D_\theta : L^2(\mathbb{T}^2_\theta)\otimes\mathbb{C}^2 \to L^2(\mathbb{T}^2_\theta)\otimes\mathbb{C}^2$ is

$$D_\theta = i(\partial_1\otimes\sigma_1 + \partial_2\otimes\sigma_2)$$

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- $(C^\infty(\mathbb{T}^2_\theta), L^2(\mathbb{T}^2_\theta) \otimes \mathbb{C}^2, D_\theta)$ is a spectral triple.

Some Applications to Physics

A Unifying Picture

NCG gives a unifying picture of the phase space and its observables in classical and quantum mechanics.

Mechanics	Phase Space	Observables
Classical	A manifold X	$f : X \to \mathbb{C}$ continuous
Quantum	H Hilbert space	Self-adjoint operators

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Mechanics	Phase Space	Observables
Classical	Comm. C^* algebra	$a \in A$
Quantum	C^*-algebra	$a \in A$

Heisenberg and Tori

Exponentiating the Heisenberg commutation relation

$$[\hat{p}, \hat{q}] = i\hbar$$

and using the Baker-Hausdorff formula it is easy to find that

$$UV = e^{i\hbar} VU \quad \textit{where } U = e^{i\hat{q}},\ V = e^{i\hat{p}}$$

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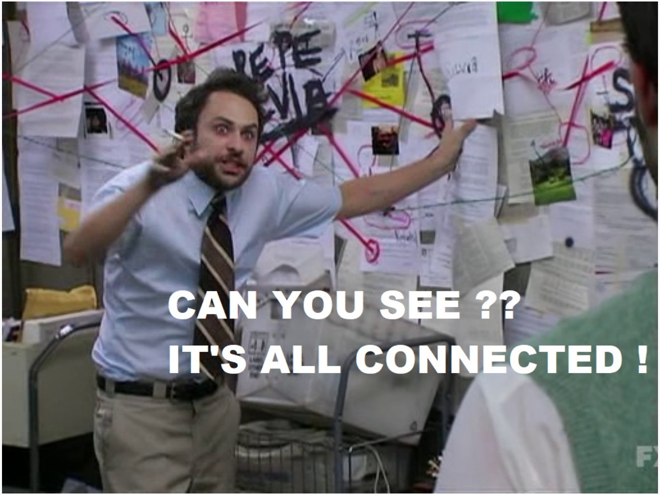
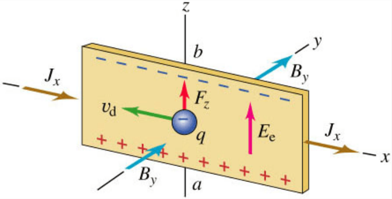
Moral of the Story

Heisenberg Commutation Relation = Noncommutative Torus

Remark

In the limit $\hbar \mapsto 0$ we get a commutative torus. This is not surprising if we remember Liuville-Arnold Theorem of classical mechanics.


CAN YOU SEE ??
IT'S ALL CONNECTED !



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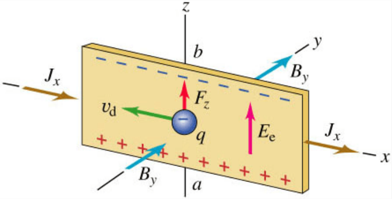

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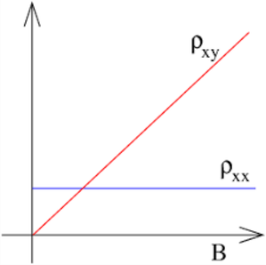
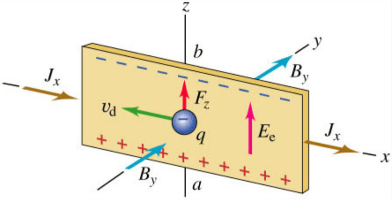


The Classical behaviour



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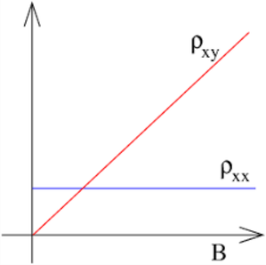

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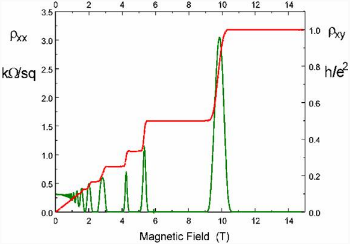
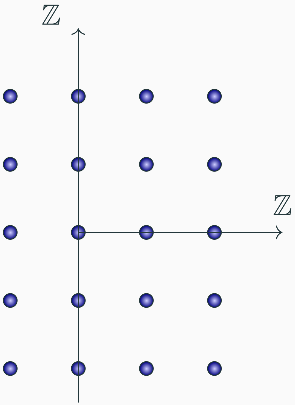


The Quantum behaviour



Let us model the Hall effect as a 2-dim lattice model with Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z}^2)$ and Hamiltonian $H = U + U^* + V + V^*$ where

$$(U\psi)(m,n) = \psi(m-1,n)$$
$$(V\psi)(m,n) = e^{-2\pi i \phi m}\psi(m,n-1)$$

and ϕ is interpreted as the magnetic flux through a unit cell.

The NC Geometry of the Hall Effect

- Note that $UV = e^{-2\pi i\phi} VU$ so that $C^*(U, V)$ is the noncommutative torus $A_{-\phi} = \mathcal{C}(\mathbb{T}^2_{-\phi})$.
- Provided that the Fermi level $\mu \notin \sigma(H)$ (that is, the system is an insulator), the Fermi projection P_μ defines a class $[P_\mu]$ in the K-theory group $K_0(A_\phi)$.
- Let X_j be the position operators $(X_j\psi)(n_1, n_2) = n_j\psi(n_1, n_2)$. One can show that

$$\left(\mathcal{C}(\mathbb{T}^2_\phi), \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^2, D = \begin{pmatrix} 0 & X_1 - iX_2 \\ X_1 + iX_2 & 0 \end{pmatrix}\right)$$

is a spectral triple. Up to "homotopy" this defines a class $[X]$ in K-homology.

The Natural Quantization of the Current

There is a perfect pairing $K_0(A_\phi) \times K^0(A_\phi) \longrightarrow \mathbb{Z}$ between K-theory and K-homology given by the Fredholm index

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Noncommutativity $\Longrightarrow$ Quantization of the Current

A Short Bibliography

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